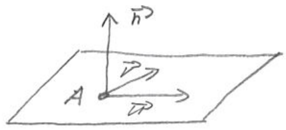
ANALYTICKÁ GEOMETRIE

ROVINA V $\mathbb{E}_3$

$\rho = [A, \vec{u}, \vec{v}]$ $\quad$ $A = [x_A, y_A, z_A]$, $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$



__parametrické vyjádření:__ $\quad X = A + t\cdot\vec{u} + s\cdot\vec{v}$

$$\begin{aligned} x &= x_A + t\cdot u_1 + s\cdot v_1 \\ y &= y_A + t\cdot u_2 + s\cdot v_2 \\ z &= z_A + t\cdot u_3 + s\cdot v_3 \end{aligned} \quad , \qquad t, s \in \mathbb{R}$$

__obecná rovnice:__ $\quad ax + by + cz + d = 0$

– normálový vektor: $\vec{n} = (a, b, c)$

$\vec{n} = k\cdot(\vec{u} \times \vec{v})$

Př: Určete obecnou rovnici roviny α: $\begin{aligned} x &= 2 + 3t - 4s \\ y &= 4 - s \\ z &= 2 + 3t \end{aligned}$

$A[2, 4, 2]$, $\vec{u} = (3, 0, 3)$, $\vec{v} = (-4, -1, 0)$

a) Užitím vektorového součinu:

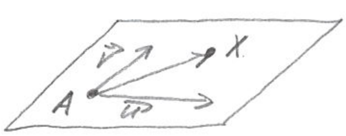
$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & 3 \\ -4 & -1 & 0 \end{vmatrix} = 3\vec{i} - 12\vec{j} - 3\vec{k} = (3, -12, -3) \Rightarrow \vec{n} = (1, -4, -1)$$

α: $x - 4y - z + d = 0$

$A \in \alpha \Rightarrow 2 - 4\cdot 4 - 2 + d = 0 \Rightarrow d = 16$

$\underline{\underline{\alpha: x - 4y - z + 16 = 0}}$

b) Užitím smíšeného součinu:



$\vec{AX}, \vec{u}, \vec{v}$ komplanární $\Rightarrow [\vec{AX}, \vec{u}, \vec{v}] = 0$

$X[x, y, z]$, $\vec{AX} = X - A = (x-2, y-4, z-2)$

$$\begin{vmatrix} x-2 & y-4 & z-2 \\ 3 & 0 & 3 \\ -4 & -1 & 0 \end{vmatrix} = (x-2)\cdot 3 - (y-4)\cdot 12 + (z-2)\cdot(-3) = 3x - 12y - 3z + 48 = 0$$

$\underline{\underline{\alpha: x - 4y - z + 16 = 0}}$

PŘÍMKA V $\mathbb{E}_3$

$p = (A, \vec{u})$ $\quad A = [x_A, y_A, z_A]$
$\vec{u} = (u_1, u_2, u_3)$ ... směrový vektor

parametrické vyjádření :
$$X = A + t \cdot \vec{u}$$
$$x = x_A + t \cdot u_1$$
$$y = y_A + t \cdot u_2$$
$$z = z_A + t \cdot u_3 \quad , \quad t \in \mathbb{R}$$

kanonický tvar : $\dfrac{x - x_A}{u_1} = \dfrac{y - y_A}{u_2} = \dfrac{z - z_A}{u_3}$ ... vznikne vyjádřením t

obecné rovnice :
- průsečnice dvou rovin

$$p: \ a_1 x + b_1 y + c_1 z + d_1 = 0 \rightarrow \vec{n}_1$$
$$a_2 x + b_2 y + c_2 z + d_2 = 0 \rightarrow \vec{n}_2$$

$\vec{n}_1 = (a_1, b_1, c_1)$
$\vec{n}_2 = (a_2, b_2, c_2)$

$\vec{u} = k \cdot (\vec{n}_1 \times \vec{n}_2)$

$\vec{u} \perp \vec{n}_1$
$\vec{u} \perp \vec{n}_2$

Př: Určete parametrické vyjádření přímky p

$$p: \ x + y - z + 5 = 0 \Rightarrow \vec{n}_1 = (1, 1, -1)$$
$$2x - y + 2z - 2 = 0 \Rightarrow \vec{n}_2 = (2, -1, 2)$$

$$\vec{u} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 2 & -1 & 2 \end{vmatrix} = \vec{i}(2-1) - \vec{j}(2+2) + \vec{k}(-1-2) = (1, -4, -3)$$

libovolný bod $A \in p$: např. $x = 0 \Rightarrow$
$$y - z + 5 = 0 \Rightarrow y = z - 5 = -8$$
$$-y + 2z - 2 = 0$$
$$z + 3 = 0 \Rightarrow z = -3$$

$A[0, -8, -3]$

$$p: \ X = A + t \cdot \vec{u}$$
$$x = t$$
$$y = -8 - 4t$$
$$z = -3 - 3t \quad , \quad t \in \mathbb{R}$$

DVĚ ROVINY

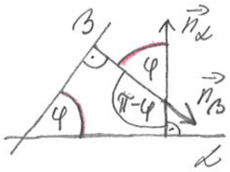
1. rovnoběžné totožné : $\vec{u}_\alpha = k\cdot\vec{u}_\beta$; lib. $A\in\alpha \Rightarrow A\in\beta$
2. rovnoběžné různé : $\vec{u}_\alpha = k\cdot\vec{u}_\beta$ $\rightarrow$ vzdálenost
3. různoběžné : $\vec{u}_\alpha \neq k\cdot\vec{u}_\beta$ $\rightarrow$ úhel

Př: Určete vzájemnou polohu rovin α a β.

1) $\alpha: 4x+2y-4z+5=0$
$\beta: 2x+y+2z-1=0$

$$\left.\begin{array}{l}\vec{n}_\alpha = (4,2,-4) = 2(2,1,-2)\\ \vec{n}_\beta = (2,1,2)\end{array}\right\} \Rightarrow \vec{n}_\alpha \neq k\cdot\vec{n}_\beta \Rightarrow \underline{\underline{\text{různoběžné}}}$$

úhel rovin = úhel normálových vektorů



$$\cos\varphi = \frac{|\vec{n}_\alpha\cdot\vec{n}_\beta|}{\|\vec{n}_\alpha\|\cdot\|\vec{n}_\beta\|}, \quad \varphi\in\langle 0, \tfrac{\pi}{2}\rangle$$

$$\cos\varphi = \frac{|8+2-8|}{\sqrt{16+4+16}\,\sqrt{4+1+4}} = \frac{2}{\sqrt{36}\cdot\sqrt{9}} = \frac{2}{6\cdot 3} = \frac{1}{9} \Rightarrow \underline{\underline{\varphi = \arccos\frac{1}{9}}}$$

2) $\alpha: x-3z+2=0$
$\beta: 2x-6z-7=0$

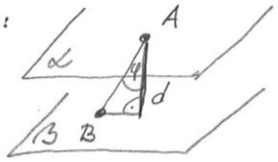
$$\left.\begin{array}{l}\vec{n}_\alpha = (1,0,-3)\\ \vec{n}_\beta = (2,0,-6)\end{array}\right\} \Rightarrow \vec{n}_\beta = 2\cdot\vec{n}_\alpha \Rightarrow \underline{\underline{\alpha \parallel \beta}}$$

$A\in\alpha$ lib: $y=0,\ z=0 \rightarrow x=-2$ ... $A[-2,0,0]$

$A\in\beta$? ... $2\cdot(-2)-6\cdot 0-7\neq 0 \Rightarrow A\notin\beta \Rightarrow \underline{\underline{\alpha\neq\beta}}$

Vzdálenost α a β = vzdálenost A od β –
– vzdálenost bodu od roviny – viz později

II. zp:



$$\left.\begin{array}{l}\cos\varphi = \dfrac{d}{\|\vec{AB}\|}\\[2mm] \cos\varphi = \dfrac{\vec{n}_\alpha\cdot\vec{AB}}{\|\vec{n}_\alpha\|\cdot\|\vec{AB}\|}\end{array}\right\} \Rightarrow d = \frac{\vec{n}_\alpha\cdot\vec{AB}}{\|\vec{n}_\alpha\|}$$

$B\in\beta$ lib: $y=0,\ z=0 \rightarrow x=\frac{7}{2}$ ... $B[\frac{7}{2},0,0]$

$\vec{AB} = (\frac{11}{2},0,0)$

$$v(\alpha,\beta) = d = \frac{\frac{11}{2}}{\sqrt{1+9}} = \frac{11}{2\sqrt{10}} = \underline{\underline{\frac{11}{20}\sqrt{10}}}$$

PŘÍMKA A ROVINA

$p = (A, \vec{u})$ $\qquad$ $\alpha: ax + by + cz + d = 0\ ,\quad \vec{n} = (a, b, c)$

1, $p \subset \alpha : \vec{u} \perp \vec{n}\ ,\ A \in \alpha$

2, $p \parallel \alpha : \vec{u} \perp \vec{n}\ ,\ A \notin \alpha$ $\Rightarrow$ vzdálenost p od α

3, p různoběžná s α : $\vec{u} \not\perp \vec{n}$ $\Rightarrow$ průsečík, úhel

Př: Určete vzájemnou polohu přímky p a roviny α.

$p: \begin{array}{l} x = 1 + t \\ y = -1 - 2t \\ z = 6t \end{array}$ $\qquad$ $\alpha: 2x + 3y + z - 1 = 0$

$\vec{u} = (1, -2, 6)\ ,\quad \vec{n} = (2, 3, 1)$

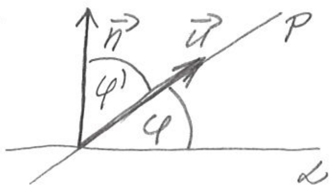
$\vec{u} \cdot \vec{n} = 2 - 6 + 6 = 2 \neq 0 \Rightarrow \vec{u} \not\perp \vec{n} \Rightarrow$ <u>p různoběžná s α</u>

<u>$p \cap \alpha$</u>:
$$\begin{aligned} 2(1+t) + 3(-1-2t) + 6t - 1 &= 0 \\ 2 + 2t - 3 - \cancel{6t} + \cancel{6t} - 1 &= 0 \\ 2t - 2 &= 0 \\ t &= 1 \end{aligned}$$

$R = p \cap \alpha$: $\begin{array}{l} x_R = 1 + 1 = 2 \\ y_R = -1 - 2 = -3 \\ z_R = 6 \end{array}$

<u>$R[2, -3, 6]$</u>

<u>$\varphi = \measuredangle(p, \alpha)$:</u>



$\varphi \in \langle 0, \frac{\pi}{2} \rangle$

$\varphi = \frac{\pi}{2} - \varphi'$

$\cos\varphi' = \dfrac{|\vec{u} \cdot \vec{n}|}{\|\vec{u}\| \cdot \|\vec{n}\|}$

$$\cos\varphi' = \frac{|2 - 6 + 6|}{\sqrt{1+4+36} \cdot \sqrt{4+9+1}} = \frac{2}{\sqrt{41} \cdot \sqrt{14}}$$

$$\underline{\varphi = \frac{\pi}{2} - \arccos\frac{2}{\sqrt{41} \cdot \sqrt{14}}}$$

DVĚ PŘÍMKY

$p = (A, \vec{u}_p)$, $q = (B, \vec{u}_q)$

1, totožné: $\vec{u}_p = k \cdot \vec{u}_q$; $p \cap q = p = q$

2, rovnoběžné různé: $\vec{u}_p = k \cdot \vec{u}_q$; $p \cap q = \emptyset$ → vzdálenost

3, různoběžné: $\vec{u}_p \neq k \cdot \vec{u}_q$; $p \cap q = R$ → průsečík, úhel

4, mimoběžné: $\vec{u}_p \neq k \cdot \vec{u}_q$; $p \cap q = \emptyset$ → vzdálenost

Př: Určete vzájemnou polohu přímek p a q.

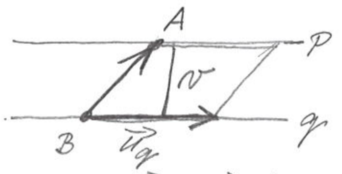
1) $p: \begin{aligned} x &= 2+3t \\ y &= -1+4t \\ z &= 2t \end{aligned}$ $\qquad q: \begin{aligned} x &= 4+3s \\ y &= 1+4s \\ z &= 3+2s \end{aligned}$

$$\left.\begin{aligned} \vec{u}_p &= (3,4,2) \\ \vec{u}_q &= (3,4,2) \end{aligned}\right\} \Rightarrow \underline{\underline{p \parallel q}}$$

$$A[2,-1,0] \in p \quad \dots \quad A \in q? : \left.\begin{aligned} 2 &= 7+3s &\Rightarrow s &= -\tfrac{5}{3} \\ -1 &= 1+4s &\Rightarrow s &= \tfrac{1}{2} \\ 0 &= 3+2s &\Rightarrow s &= -\tfrac{3}{2} \end{aligned}\right\} \Rightarrow A \notin q \Rightarrow$$

$$\Rightarrow \underline{\underline{p \neq q}}$$

vzdálenost:



$B[7,1,3] \in q$

$\vec{AB} = (5, 2, 3)$

$$v(p,q) = v = \frac{\| \vec{AB} \times \vec{u}_q \|}{\| \vec{u}_q \|} = \frac{3\sqrt{29}}{\sqrt{29}} = \underline{\underline{3}}$$

$$\vec{AB} \times \vec{u}_q = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 2 & 3 \\ 3 & 4 & 2 \end{vmatrix} = \vec{i} \cdot (4-12) - \vec{j} \cdot (10-9) + \vec{k} \cdot (20-6) = (-8, -1, 14)$$

$$\| \vec{AB} \times \vec{u}_q \| = \sqrt{64+1+196} = \sqrt{261} = 3\sqrt{29}$$

$$\| \vec{u}_q \| = \sqrt{9+16+4} = \sqrt{29}$$

2) $p: \begin{cases} x = -1+3t \\ y = -7+2t \\ z = 4-t \end{cases}$ $\qquad$ $q: \begin{cases} x = 2+3s \\ y = -5-2s \\ z = 3-s \end{cases}$

$\vec{u}_p = (3,2,-1)$ $\qquad$ $\vec{u}_q = (3,-2,-1)$

$A[-1,-7,4]$ $\qquad$ $B = [2,-5,3]$ $\qquad$ $\overrightarrow{AB} = (3,2,-1)$

$\vec{u}_p \neq k\cdot\vec{u}_q \Rightarrow p \nparallel q$

$\vec{u}_p, \vec{u}_q, \overrightarrow{AB}$ komplanární $\Rightarrow$ p, q leží v jedné rovině

$$[\vec{u}_p, \vec{u}_q, \overrightarrow{AB}] = \begin{vmatrix} 3 & 2 & -1 \\ 3 & -2 & -1 \\ 3 & 2 & -1 \end{vmatrix} = 0 \Rightarrow p, q \text{ \underline{různoběžné}}$$

$\underline{p \cap q = R}$:

$$\begin{aligned} -1+3t &= 2+3s &\Rightarrow 3t-3s &= 3 \quad |:3 \\ -7+2t &= -5-2s &\Rightarrow 2t+2s &= 2 \quad |:2 \\ 4-t &= 3-s &\Rightarrow -t+s &= -1 \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{aligned} t-s &= 1 \\ 2s &= 0 \end{aligned} \Rightarrow \begin{aligned} t &= 1 \\ s &= 0 \end{aligned}$$

$\underline{\underline{R[2,-5,3]}}$

$$\underline{\varphi = \measuredangle(p,q)}: \quad \cos\varphi = \frac{|\vec{u}_p \cdot \vec{u}_q|}{\|\vec{u}_p\| \cdot \|\vec{u}_q\|} = \frac{|9-4+1|}{\sqrt{14}\cdot\sqrt{14}} = \frac{6}{14} = \frac{3}{7} \Rightarrow \underline{\underline{\varphi = \arccos\frac{3}{7}}}$$

3) $p: \begin{cases} x = t-1 \\ y = t \\ z = 2t+1 \end{cases}$ $\qquad$ $q: \begin{cases} x = s \\ y = 3s-1 \\ z = 4s+2 \end{cases}$

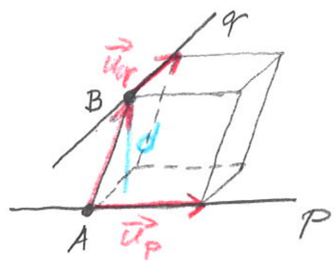
$\vec{u}_p = (1,1,2)$, $A = [-1,0,1]$ $\qquad$ $\vec{u}_q = (1,3,4)$, $B[0,-1,2]$

$\vec{u}_p \neq k\cdot\vec{u}_q \Rightarrow p \nparallel q$ $\qquad$ $\overrightarrow{AB} = (1,-1,1)$

$$[\vec{u}_p, \vec{u}_q, \overrightarrow{AB}] = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & -1 & 1 \end{vmatrix} \cdot(-1) = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 2 \neq 0 \Rightarrow$$

$\Rightarrow p, q$ **mimoběžné**

$\underline{d = v(p,q)}$ ... výška rovnoběžnostěnu



$$V = S\cdot d \Rightarrow d = \frac{V}{S} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} = \underline{\underline{\frac{\sqrt{3}}{3}}}$$

$V = |[\vec{u}_p, \vec{u}_q, \overrightarrow{AB}]| = 2$

$$\vec{u}_p \times \vec{u}_q = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ 1 & 3 & 4 \end{vmatrix} = \vec{i}(4-6) - \vec{j}(4-2) + \vec{k}(3-1) = (-2,-2,2)$$

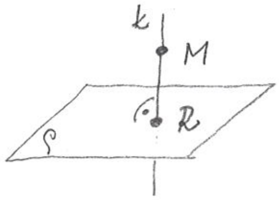
$S = \|\vec{u}_p \times \vec{u}_q\| = \sqrt{4+4+4} = 2\sqrt{3}$

METRICKÉ ÚLOHY

- vzdálenosti; kolmost

Př: Je dán bod M a rovina ρ. Určete:
a) Kolmý průmět bodu M do roviny ρ.
b) Vzdálenost bodu M od roviny ρ.

$M[-2, 0, 8]$ $\quad \rho: -x - y + 2z = 0$



1. $k;\ M \in k \perp \rho$
2. $R;\ R \in k \cap \rho \rightarrow a)$
3. $d = v|M, \rho| = |MR| \rightarrow b)$

1. $k \perp \rho \Rightarrow \vec{u}_k = \vec{n}_\rho = (-1, -1, 2) \rightarrow k: X = M + t \cdot \vec{u}_k$

$$x = -2 - t$$
$$y = -t$$
$$z = 8 + 2t$$

2. $-(-2-t) - (-t) + 2(8+2t) = 0$

$$2 + t + t + 16 + 4t = 0$$
$$6t = -18$$
$$t = -3$$

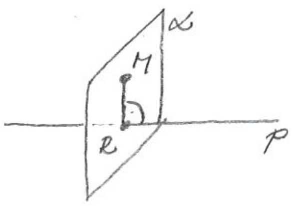
$x_R = -2 - (-3) = 1$
$y_R = -(-3) = 3$
$z_R = 8 + 2 \cdot (-3) = 2$

$R[1, 3, 2]$

3. $d = v|M, \rho| = |MR| = \sqrt{(1-(-2))^2 + (3-0)^2 + (2-8)^2} = \sqrt{9+9+36} = 3\sqrt{6}$

Př: Je dán bod M a přímka p. Určete:
a) Kolmý průmět bodu M na přímku p.
b) Vzdálenost bodu M od přímky p.

$M[7, -3, 3]$ $\quad p \equiv \{x = 1 + t,\ y = -3 + 2t,\ z = -3 + 2t\}$



1. $\alpha;\ M \in \alpha \perp p$
2. $R;\ R \in \alpha \cap p \rightarrow a)$
3. $d = v|M, p| = |MR| \rightarrow b)$

1. $\alpha \perp p \Rightarrow \vec{n}_\alpha = \vec{u}_p = (1, 2, 2) \rightarrow \alpha: x + 2y + 2z + d = 0$

$M \in \alpha \Rightarrow 7 + 2 \cdot (-3) + 2 \cdot 3 + d = 0 \Rightarrow d = -7$

$\alpha: x + 2y + 2z - 7 = 0$

2. $(1+t) + 2(-3+2t) + 2(-3+2t) - 7 = 0$

$$1 + t - 6 + 4t - 6 + 4t - 7 = 0$$
$$9t = 18$$
$$t = 2$$

$x_R = 1 + 2 = 3$
$y_R = -3 + 2 \cdot 2 = 1$
$z_R = -3 + 2 \cdot 2 = 1$

$R[3, 1, 1]$

3. $d = v|M, p| = |MR| = \sqrt{(3-7)^2 + (1+3)^2 + (1-3)^2} = \sqrt{16+16+4} = \sqrt{36} = 6$